

CHAPTER 11

Fourier Series

Fourier series were first formalized by the French mathematician Jean-Baptiste Joseph Fourier (1768–1830) as a result of his work on solving a particular partial differential equation known as the heat conduction equation. However, the actual introduction of the so-called Fourier theory was motivated by a problem in musical acoustics concerning vibrating strings. Daniel Bernoulli (1700–1782) is credited as being the first to model the motion of a vibrating string as a series of trigonometric functions in 1748, twenty years before the birth of Fourier. The actual development of Fourier theory took place in 1807 upon Fourier's return from Egypt, where he was a participant in the Egyptian campaign of 1798 under Napoleon Bonaparte.

11.1. INTRODUCTION

A series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad (11.1)$$

is called a trigonometric series. Let $f(x)$ be a function defined and Riemann integrable on the interval $[-\pi, \pi]$. By definition, the Fourier series associated with $f(x)$ is a trigonometric series of the form (11.1), where a_n and b_n are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = 0, 1, 2, \dots, \quad (11.2)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \dots. \quad (11.3)$$

In this case, we write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]. \quad (11.4)$$

The numbers a_n and b_n are called the Fourier coefficients of $f(x)$. The symbol $\sim$ is used here instead of equality because at this stage, nothing is known about the convergence of the series in (11.4) for all x in $[-\pi, \pi]$. Even if the series converges, it may not converge to $f(x)$.

We can also consider the following reverse approach: if the trigonometric series in (11.4) is uniformly convergent to $f(x)$ on $[-\pi, \pi]$, that is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx], \quad (11.5)$$

then a_n and b_n are given by formulas (11.2) and (11.3). In this case, the derivation of a_n and b_n is obtained by multiplying both sides of (11.5) by $\cos nx$ and $\sin nx$, respectively, followed by integration over $[-\pi, \pi]$. More specifically, to show formula (11.2), we multiply both sides of (11.5) by $\cos nx$. For $n \neq 0$, we then have

$$f(x) \cos nx = \frac{a_0}{2} \cos nx + \sum_{k=1}^{\infty} [a_k \cos kx \cos nx + b_k \sin kx \cos nx] dx. \quad (11.6)$$

Since the series on the right-hand side converges uniformly, it can be integrated term by term (this can be easily proved by applying Theorem 6.6.1 to the sequence whose n th term is the n th partial sum of the series). We then have

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos nx dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos nx dx \\ &+ \sum_{k=1}^{\infty} \left[\int_{-\pi}^{\pi} a_k \cos kx \cos nx dx + \int_{-\pi}^{\pi} b_k \sin kx \cos nx dx \right]. \end{aligned} \quad (11.7)$$

But

$$\int_{-\pi}^{\pi} \cos nx dx = 0, \quad n = 1, 2, \dots, \quad (11.8)$$

$$\int_{-\pi}^{\pi} \sin kx \cos nx dx = 0, \quad (11.9)$$

$$\int_{-\pi}^{\pi} \cos kx \cos nx dx = \begin{cases} 0, & k \neq n, \\ \pi, & k = n \geq 1. \end{cases} \quad (11.10)$$

From (11.7)–(11.10) we conclude (11.2). Note that formulas (11.9) and (11.10) can be shown to be true by recalling the following trigonometric identities:

$$\begin{aligned}\sin kx \cos nx &= \frac{1}{2} \{ \sin[(k+n)x] + \sin[(k-n)x] \}, \\ \cos kx \cos nx &= \frac{1}{2} \{ \cos[(k+n)x] + \cos[(k-n)x] \}.\end{aligned}$$

For $n = 0$ we obtain from (11.7),

$$\int_{-\pi}^{\pi} f(x) dx = a_0 \pi.$$

Formula (11.3) for b_n can be proved similarly. We can therefore state the following conclusion: a uniformly convergent trigonometric series is the Fourier series of its sum.

Note. If the series in (11.1) converges or diverges at a point x_0 , then it converges or diverges at $x_0 + 2n\pi$ ($n = 1, 2, \dots$) due to the periodic nature of the sine and cosine functions. Thus, if the series (11.1) represents a function $f(x)$ on $[-\pi, \pi]$, then the series also represents the so-called periodic extension of $f(x)$ for all values of x . Geometrically speaking, the periodic extension of $f(x)$ is obtained by shifting the graph of $f(x)$ on $[-\pi, \pi]$ by $2\pi, 4\pi, \dots$ to the right and to the left. For example, for $-3\pi < x \leq -\pi$, $f(x)$ is defined by $f(x + 2\pi)$, and for $\pi < x \leq 3\pi$, $f(x)$ is defined by $f(x - 2\pi)$, etc. This defines $f(x)$ for all x as a periodic function with period 2π .

EXAMPLE 11.1.1. Let $f(x)$ be defined on $[-\pi, \pi]$ by the formula

$$f(x) = \begin{cases} 0, & -\pi \leq x < 0, \\ \frac{x}{\pi}, & 0 \leq x \leq \pi. \end{cases}$$

Then, from (11.2) we have

$$\begin{aligned}a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= \frac{1}{\pi^2} \int_0^{\pi} x \cos nx dx \\ &= \frac{1}{\pi^2} \left[\frac{x \sin nx}{n} \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin nx dx \right] \\ &= \frac{1}{\pi^2 n^2} [(-1)^n - 1].\end{aligned}$$

Thus, for $n = 1, 2, \dots$,

$$a_n = \begin{cases} 0, & n \text{ even,} \\ -2/(\pi^2 n^2), & n \text{ odd.} \end{cases}$$

Also, from (11.3), we get

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{1}{\pi^2} \int_0^{\pi} x \sin nx \, dx \\ &= \frac{1}{\pi^2} \left[-\frac{x \cos nx}{n} \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right] \\ &= \frac{1}{\pi^2} \left[-\frac{\pi}{n} (-1)^n + \frac{\sin nx}{n^2} \Big|_0^{\pi} \right] \\ &= \frac{(-1)^{n+1}}{\pi n}. \end{aligned}$$

For $n = 0$, a_0 is given by

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx \\ &= \frac{1}{\pi^2} \int_0^{\pi} x \, dx \\ &= \frac{1}{2}. \end{aligned}$$

The Fourier series of $f(x)$ is then of the form

$$f(x) \sim \frac{1}{4} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos[(2n-1)x] - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx.$$

EXAMPLE 11.1.2. Let $f(x) = x^2$ ($-\pi \leq x \leq \pi$). Then

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx \\ &= \frac{2\pi^2}{3}, \end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx \\
&= \frac{x^2 \sin nx}{\pi n} \Big|_{-\pi}^{\pi} - \frac{2}{\pi n} \int_{-\pi}^{\pi} x \sin nx \, dx \\
&= -\frac{2}{\pi n} \int_{-\pi}^{\pi} x \sin nx \, dx \\
&= \frac{2x \cos nx}{\pi n^2} \Big|_{-\pi}^{\pi} - \frac{2}{\pi n^2} \int_{-\pi}^{\pi} \cos nx \, dx \\
&= \frac{4 \cos n\pi}{n^2} \\
&= \frac{4(-1)^n}{n^2}, \quad n = 1, 2, \dots, \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx \, dx \\
&= 0,
\end{aligned}$$

since $x^2 \sin nx$ is an odd function. Thus, the Fourier expansion of $f(x)$ is

$$x^2 \sim \frac{\pi^2}{3} - 4 \left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right).$$

11.2. CONVERGENCE OF FOURIER SERIES

In this section, we consider the conditions under which the Fourier series of $f(x)$ converges to $f(x)$. We shall assume that $f(x)$ is Riemann integrable on $[-\pi, \pi]$. Hence, $f^2(x)$ is also Riemann integrable on $[-\pi, \pi]$ by Corollary 6.4.1. This condition will be satisfied if, for example, $f(x)$ is continuous on $[-\pi, \pi]$, or if it has a finite number of discontinuities of the first kind (see Definition 3.4.2) in this interval.

In order to study the convergence of Fourier series, the following lemmas are needed:

Lemma 11.2.1. If $f(x)$ is Riemann integrable on $[-\pi, \pi]$, then

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0, \quad (11.11)$$

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0. \quad (11.12)$$

Proof. Let $s_n(x)$ denote the following partial sum of Fourier series of $f(x)$:

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^n [a_k \cos kx + b_k \sin kx], \quad (11.13)$$

where a_k ($k = 0, 1, \dots, n$) and b_k ($k = 1, 2, \dots, n$) are given by (11.2) and (11.3), respectively. Then

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) s_n(x) dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} f(x) dx \\ &\quad + \sum_{k=1}^n \left[a_k \int_{-\pi}^{\pi} f(x) \cos kx dx + b_k \int_{-\pi}^{\pi} f(x) \sin kx dx \right] \\ &= \frac{\pi a_0^2}{2} + \pi \sum_{k=1}^n (a_k^2 + b_k^2). \end{aligned}$$

It can also be verified that

$$\int_{-\pi}^{\pi} s_n^2(x) dx = \frac{\pi a_0^2}{2} + \pi \sum_{k=1}^n (a_k^2 + b_k^2).$$

Consequently,

$$\begin{aligned} \int_{-\pi}^{\pi} [f(x) - s_n(x)]^2 dx &= \int_{-\pi}^{\pi} f^2(x) dx - 2 \int_{-\pi}^{\pi} f(x) s_n(x) dx + \int_{-\pi}^{\pi} s_n^2(x) dx \\ &= \int_{-\pi}^{\pi} f^2(x) dx - \left[\frac{\pi a_0^2}{2} + \pi \sum_{k=1}^n (a_k^2 + b_k^2) \right]. \end{aligned}$$

It follows that

$$\frac{a_0^2}{2} + \sum_{k=1}^n (a_k^2 + b_k^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx. \quad (11.14)$$

Since the right-hand side of (11.14) exists and is independent of n , the series $\sum_{k=1}^{\infty} (a_k^2 + b_k^2)$ must be convergent. This follows from applying Theorem 5.1.2 and the fact that the sequence

$$s_n^* = \sum_{k=1}^n (a_k^2 + b_k^2)$$

is bounded and monotone increasing. But, the convergence of $\sum_{k=1}^{\infty} (a_k^2 + b_k^2)$ implies that $\lim_{k \rightarrow \infty} (a_k^2 + b_k^2) = 0$ (see Result 5.2.1 in Chapter 5). Hence, $\lim_{k \rightarrow \infty} a_k = 0$, $\lim_{k \rightarrow \infty} b_k = 0$. $\square$

Corollary 11.2.1. If $\phi(x)$ is Riemann integrable on $[-\pi, \pi]$, then

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} \phi(x) \sin\left[\left(n + \frac{1}{2}\right)x\right] dx = 0. \quad (11.15)$$

Proof. We have that

$$\phi(x) \sin\left[\left(n + \frac{1}{2}\right)x\right] = \left[\phi(x) \cos \frac{x}{2}\right] \sin nx + \left[\phi(x) \sin \frac{x}{2}\right] \cos nx.$$

Let $\phi_1(x) = \phi(x) \sin(x/2)$, $\phi_2(x) = \phi(x) \cos(x/2)$. Both $\phi_1(x)$ and $\phi_2(x)$ are Riemann integrable by Corollary 6.4.2. By applying Lemma 11.2.1 to both $\phi_1(x)$ and $\phi_2(x)$, we obtain

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} \phi_1(x) \cos nx dx = 0, \quad (11.16)$$

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} \phi_2(x) \sin nx dx = 0. \quad (11.17)$$

Formula (11.15) follows from the addition of (11.16) and (11.17). $\square$

Corollary 11.2.2. If $\phi(x)$ is Riemann integrable on $[-\pi, \pi]$, then

$$\lim_{n \rightarrow \infty} \int_{-\pi}^0 \phi(x) \sin\left[\left(n + \frac{1}{2}\right)x\right] dx = 0,$$

$$\lim_{n \rightarrow \infty} \int_0^{\pi} \phi(x) \sin\left[\left(n + \frac{1}{2}\right)x\right] dx = 0.$$

Proof. Define the functions $h_1(x)$ and $h_2(x)$ as

$$h_1(x) = \begin{cases} 0, & 0 \leq x \leq \pi, \\ \phi(x), & -\pi \leq x < 0, \end{cases}$$

$$h_2(x) = \begin{cases} \phi(x), & 0 \leq x \leq \pi, \\ 0, & -\pi \leq x < 0. \end{cases}$$

Both $h_1(x)$ and $h_2(x)$ are Riemann integrable on $[-\pi, \pi]$. Hence, by Corollary 11.2.1,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{-\pi}^0 \phi(x) \sin\left[\left(n + \frac{1}{2}\right)x\right] dx &= \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} h_1(x) \sin\left[\left(n + \frac{1}{2}\right)x\right] dx \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^{\pi} \phi(x) \sin\left[\left(n + \frac{1}{2}\right)x\right] dx &= \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} h_2(x) \sin\left[\left(n + \frac{1}{2}\right)x\right] dx \\ &= 0. \quad \square \end{aligned}$$

Lemma 11.2.2.

$$\frac{1}{2} + \sum_{k=1}^n \cos ku = \frac{\sin[(n + \frac{1}{2})u]}{2 \sin(u/2)}. \quad (11.18)$$

Proof. Let $G_n(u)$ be defined as

$$G_n(u) = \frac{1}{2} + \sum_{k=1}^n \cos ku.$$

Multiplying both sides by $2 \sin(u/2)$ and using the identity

$$2 \sin \frac{u}{2} \cos ku = \sin[(k + \frac{1}{2})u] - \sin[(k - \frac{1}{2})u],$$

we obtain

$$\begin{aligned} 2 \sin \frac{u}{2} G_n(u) &= \sin \frac{u}{2} + \sum_{k=1}^n \{\sin[(k + \frac{1}{2})u] - \sin[(k - \frac{1}{2})u]\} \\ &= \sin[(n + \frac{1}{2})u]. \end{aligned}$$

Hence, if $\sin(u/2) \neq 0$, then

$$G_n(u) = \frac{\sin[(n + \frac{1}{2})u]}{2 \sin(u/2)}. \quad \square$$

Theorem 11.2.1. Let $f(x)$ be Riemann integrable on $[-\pi, \pi]$, and let it be extended periodically outside this interval. Suppose that at a point x , $f(x)$ satisfies the following two conditions:

- i. Both $f(x^-)$ and $f(x^+)$ exist, where $f(x^-)$ and $f(x^+)$ are the left-sided and right-sided limits of $f(x)$, and

$$f(x) = \frac{1}{2} [f(x^-) + f(x^+)]. \quad (11.19)$$

- ii. Both one-sided derivatives,

$$\begin{aligned} f'(x^+) &= \lim_{h \rightarrow 0^+} \frac{(x+h) - f(x^+)}{h}, \\ f'(x^-) &= \lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x^-)}{h}, \end{aligned}$$

exist.

Then the Fourier series of $f(x)$ converges to $f(x)$ at x , that is,

$$s_n(x) \rightarrow \begin{cases} f(x) & \text{if } x \text{ is a point of continuity,} \\ \frac{1}{2}[f(x^+) + f(x^-)] & \text{if } x \text{ is a point of discontinuity of the first kind.} \end{cases}$$

Before proving this theorem, it should be noted that if $f(x)$ is continuous at x , then condition (11.19) is satisfied. If, however, x is a point of discontinuity of $f(x)$ of the first kind, then $f(x)$ is defined to be equal to the right-hand side of (11.19). Such a definition of $f(x)$ does not affect the values of the Fourier coefficients, a_n and b_n , in (11.2) and (11.3). Hence, the Fourier series of $f(x)$ remains unchanged.

Proof of Theorem 11.2.1. From (11.2), (11.3), and (11.13), we have

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{k=1}^n \left[\frac{1}{\pi} \left(\int_{-\pi}^{\pi} f(t) \cos kt dt \right) \cos kx \right. \\ &\quad \left. + \frac{1}{\pi} \left(\int_{-\pi}^{\pi} f(t) \sin kt dt \right) \sin kx \right] \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^n (\cos kt \cos kx + \sin kt \sin kx) \right] dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^n \cos k(t-x) \right] dt. \end{aligned}$$

Using Lemma 11.2.2, $s_n(x)$ can be written as

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left\{ \frac{\sin[(n + \frac{1}{2})(t-x)]}{2 \sin[(t-x)/2]} \right\} dt. \quad (11.20)$$

If we make the change of variable $t-x=u$ in (11.20), we obtain

$$s_n(x) = \frac{1}{\pi} \int_{-\pi-x}^{\pi-x} f(x+u) \frac{\sin[(n + \frac{1}{2})u]}{2 \sin(u/2)} du.$$

Since both $f(x+u)$ and $\sin[(n + \frac{1}{2})u]/[2\sin(u/2)]$ have period 2π with respect to u , the integral from $-\pi-x$ to $\pi-x$ has the same value as the one from $-\pi$ to π . Thus,

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u) \frac{\sin[(n + \frac{1}{2})u]}{2 \sin(u/2)} du. \quad (11.21)$$

We now need to show that for each x ,

$$\lim_{n \rightarrow \infty} s_n(x) = \frac{1}{2} [f(x^-) + f(x^+)].$$

Formula (11.21) can be written as

$$\begin{aligned} s_n(x) &= \frac{1}{\pi} \int_{-\pi}^0 f(x+u) \frac{\sin[(n + \frac{1}{2})u]}{2 \sin(u/2)} du \\ &\quad + \frac{1}{\pi} \int_0^{\pi} f(x+u) \frac{\sin[(n + \frac{1}{2})u]}{2 \sin(u/2)} du \\ &= \frac{1}{\pi} \int_{-\pi}^0 [f(x+u) - f(x^-)] \frac{\sin[(n + \frac{1}{2})u]}{2 \sin(u/2)} du \\ &\quad + f(x^-) \frac{1}{\pi} \int_{-\pi}^0 \frac{\sin[(n + \frac{1}{2})u]}{2 \sin(u/2)} du \\ &\quad + \frac{1}{\pi} \int_0^{\pi} [f(x+u) - f(x^+)] \frac{\sin[(n + \frac{1}{2})u]}{2 \sin(u/2)} du \\ &\quad + f(x^+) \frac{1}{\pi} \int_0^{\pi} \frac{\sin[(n + \frac{1}{2})u]}{2 \sin(u/2)} du. \end{aligned} \quad (11.22)$$

The first integral in (11.22) can be expressed as

$$\begin{aligned} &\frac{1}{\pi} \int_{-\pi}^0 [f(x+u) - f(x^-)] \frac{\sin[(n + \frac{1}{2})u]}{2 \sin(u/2)} du \\ &= \frac{1}{\pi} \int_{-\pi}^0 \frac{f(x+u) - f(x^-)}{u} \frac{u}{2 \sin(u/2)} \sin[(n + \frac{1}{2})u] du. \end{aligned}$$

We note that the function

$$\frac{f(x+u) - f(x^-)}{u} \cdot \frac{u}{2 \sin(u/2)}$$

is Riemann integrable on $[-\pi, 0]$, and at $u = 0$ it has a discontinuity of the first kind, since

$$\begin{aligned} \lim_{u \rightarrow 0^-} \frac{f(x+u) - f(x^-)}{u} &= f'(x^-), \quad \text{and} \\ \lim_{u \rightarrow 0^-} \frac{u}{2 \sin(u/2)} &= 1, \end{aligned}$$

that is, both limits are finite. Consequently, by applying Corollary 11.2.2 to the function $\{[f(x+u) - f(x^-)]/u\}$, $u/[2\sin(u/2)]$, we get

$$\lim_{n \rightarrow \infty} \frac{1}{\pi} \int_{-\pi}^0 [f(x+u) - f(x^-)] \frac{\sin[(n + \frac{1}{2})u]}{2\sin(u/2)} du = 0. \quad (11.23)$$

We can similarly show that the third integral in (11.22) has a limit equal to zero as $n \rightarrow \infty$, that is,

$$\lim_{n \rightarrow \infty} \frac{1}{\pi} \int_0^{\pi} [f(x+u) - f(x^+)] \frac{\sin[(n + \frac{1}{2})u]}{2\sin(u/2)} du = 0. \quad (11.24)$$

Furthermore, from Lemma 11.2.2, we have

$$\begin{aligned} \int_{-\pi}^0 \frac{\sin[(n + \frac{1}{2})u]}{2\sin(u/2)} du &= \int_{-\pi}^0 \left[\frac{1}{2} + \sum_{k=1}^n \cos ku \right] du \\ &= \frac{\pi}{2}, \end{aligned} \quad (11.25)$$

$$\begin{aligned} \int_0^{\pi} \frac{\sin[(n + \frac{1}{2})u]}{2\sin(u/2)} du &= \int_0^{\pi} \left[\frac{1}{2} + \sum_{k=1}^n \cos ku \right] du \\ &= \frac{\pi}{2}. \end{aligned} \quad (11.26)$$

From (11.22)–(11.26), we conclude that

$$\lim_{n \rightarrow \infty} s_n(x) = \frac{1}{2} [f(x^-) + f(x^+)]. \quad \square$$

Definition 11.2.1. A function $f(x)$ is said to be piecewise continuous on $[a, b]$ if it is continuous on $[a, b]$ except for a finite number of discontinuities of the first kind in $[a, b]$, and, in addition, both $f(a^+)$ and $f(b^-)$ exist. $\square$

Corollary 11.2.3. Suppose that $f(x)$ is piecewise continuous on $[-\pi, \pi]$, and that it can be extended periodically outside this interval. In addition, if, at each interior point of $[-\pi, \pi]$, $f'(x^+)$ and $f'(x^-)$ exist and $f'(-\pi^+)$ and $f'(\pi^-)$ exist, then at a point x , the Fourier series of $f(x)$ converges to $\frac{1}{2}[f(x^-) + f(x^+)]$.

Proof. This follows directly from applying Theorem 11.2.1 and the fact that a piecewise continuous function on $[a, b]$ is Riemann integrable there. $\square$

EXAMPLE 11.2.1. Let $f(x) = x$ ($-\pi \leq x \leq \pi$). The periodic extension of $f(x)$ (outside the interval $[-\pi, \pi]$) is defined everywhere. In this case,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \, dx = 0, \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx \\ &= \frac{2(-1)^{n+1}}{n}, \quad n = 1, 2, \dots \end{aligned}$$

Hence, the Fourier series of $f(x)$ is

$$x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx.$$

This series converges to x at each x in $(-\pi, \pi)$. At $x = -\pi, \pi$ we have discontinuities of the first kind for the periodic extension of $f(x)$. Hence, at $x = \pi$, the Fourier series converges to

$$\begin{aligned} \frac{1}{2} [f(\pi^-) + f(\pi^+)] &= \frac{1}{2} [\pi + (-\pi)] \\ &= 0. \end{aligned}$$

Similarly, at $x = -\pi$, the series converges to

$$\begin{aligned} \frac{1}{2} [f(-\pi^-) + f(-\pi^+)] &= \frac{1}{2} [\pi + (-\pi)] \\ &= 0. \end{aligned}$$

For other values of x , the series converges to the value of the periodic extension of $f(x)$.

EXAMPLE 11.2.2. Consider the function $f(x) = x^2$ defined in Example 11.1.2. Its Fourier series is

$$x^2 \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

The periodic extension of $f(x)$ is continuous everywhere. We can therefore write

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

In particular, for $x = \mp \pi$, we have

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2},$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

11.3. DIFFERENTIATION AND INTEGRATION OF FOURIER SERIES

In Section 11.2 conditions were given under which a function $f(x)$ defined on $[-\pi, \pi]$ is represented as a Fourier series. In this section, we discuss the conditions under which the series can be differentiated or integrated term by term.

Theorem 11.3.1. Let $a_0/2 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$ be the Fourier series of $f(x)$. If $f(x)$ is continuous on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, and $f'(x)$ is piecewise continuous on $[-\pi, \pi]$, then

- a. at each point where $f''(x)$ exists, $f'(x)$ can be represented by the derivative of the Fourier series of $f(x)$, where differentiation is done term by term, that is,

$$f'(x) = \sum_{n=1}^{\infty} [nb_n \cos nx - na_n \sin nx];$$

- b. the Fourier series of $f(x)$ converges uniformly and absolutely to $f(x)$ on $[-\pi, \pi]$.

Proof.

- a. The Fourier series of $f(x)$ converges to $f(x)$ by Corollary 11.2.3. Thus,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx].$$

The periodic extension of $f(x)$ is continuous, since $f(\pi) = f(-\pi)$. Furthermore, the derivative, $f'(x)$, of $f(x)$ satisfies the conditions of Corollary 11.2.3. Hence, the Fourier series of $f'(x)$ converges to $f'(x)$, that is,

$$f'(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} [\alpha_n \cos nx + \beta_n \sin nx], \quad (11.27)$$

where

$$\begin{aligned}
 \alpha_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) dx \\
 &= \frac{1}{\pi} [f(\pi) - f(-\pi)] \\
 &= 0, \\
 \alpha_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos nx dx \\
 &= \frac{1}{\pi} f(x) \cos nx \Big|_{-\pi}^{\pi} + \frac{n}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\
 &= \frac{(-1)^n}{n} [f(\pi) - f(-\pi)] + nb_n \\
 &= nb_n, \\
 \beta_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin nx dx \\
 &= \frac{1}{\pi} f(x) \sin nx \Big|_{-\pi}^{\pi} - \frac{n}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\
 &= -na_n.
 \end{aligned}$$

By substituting α_0 , α_n , and β_n in (11.27), we obtain

$$f'(x) = \sum_{n=1}^{\infty} [nb_n \cos nx - na_n \sin nx].$$

- b.** Consider the Fourier series of $f'(x)$ in (11.27), where $\alpha_0 = 0$, $\alpha_n = nb_n$, $\beta_n = -na_n$. Then, using inequality (11.14), we obtain

$$\sum_{n=1}^{\infty} (\alpha_n^2 + \beta_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} [f'(x)]^2 dx. \quad (11.28)$$

Inequality (11.28) indicates that $\sum_{n=1}^{\infty} (\alpha_n^2 + \beta_n^2)$ is a convergent series. Now, let $s_n(x) = a_0/2 + \sum_{k=1}^n [a_k \cos kx + b_k \sin kx]$. Then, for $n \geq m+1$,

$$\begin{aligned}
 |s_n(x) - s_m(x)| &= \left| \sum_{k=m+1}^n [a_k \cos kx + b_k \sin kx] \right| \\
 &\leq \sum_{k=m+1}^n |a_k \cos kx + b_k \sin kx|.
 \end{aligned}$$

Note that

$$|a_k \cos kx + b_k \sin kx| \leq (a_k^2 + b_k^2)^{1/2}. \quad (11.29)$$

Inequality (11.29) follows from the fact that $a_k \cos kx + b_k \sin kx$ is the dot product, $\mathbf{u} \cdot \mathbf{v}$, of the vectors $\mathbf{u} = (a_k, b_k)'$, $\mathbf{v} = (\cos kx, \sin kx)'$, and $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$ (see Theorem 2.1.2). Hence,

$$\begin{aligned} |s_n(x) - s_m(x)| &\leq \sum_{k=m+1}^n (a_k^2 + b_k^2)^{1/2} \\ &= \sum_{k=m+1}^n \frac{1}{k} (\alpha_k^2 + \beta_k^2)^{1/2}. \end{aligned} \quad (11.30)$$

But, by the Cauchy-Schwarz inequality,

$$\sum_{k=m+1}^n \frac{1}{k} (\alpha_k^2 + \beta_k^2)^{1/2} \leq \left[\sum_{k=m+1}^n \frac{1}{k^2} \right]^{1/2} \left[\sum_{k=m+1}^n (\alpha_k^2 + \beta_k^2) \right]^{1/2},$$

and by (11.28),

$$\sum_{k=m+1}^n (\alpha_k^2 + \beta_k^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} [f'(x)]^2 dx.$$

In addition, $\sum_{k=1}^{\infty} 1/k^2$ is a convergent series. Hence, by the Cauchy criterion, for a given $\epsilon > 0$, there exists a positive integer N such that $\sum_{k=m+1}^n 1/k^2 < \epsilon^2$ if $n > m > N$. Hence,

$$\begin{aligned} |s_n(x) - s_m(x)| &\leq \sum_{k=m+1}^n |a_k \cos kx + b_k \sin kx| \\ &\leq c\epsilon, \quad \text{if } m > n > N, \end{aligned} \quad (11.31)$$

where $c = \{(1/\pi) \int_{-\pi}^{\pi} [f'(x)]^2 dx\}^{1/2}$. The double inequality (11.31) shows that the Fourier series of $f(x)$ converges absolutely and uniformly to $f(x)$ on $[-\pi, \pi]$ by the Cauchy criterion. $\square$

Note that from (11.30) we can also conclude that $\sum_{k=1}^{\infty} (a_k^2 + b_k^2)^{1/2}$ satisfies the Cauchy criterion. This series is therefore convergent. Furthermore, it is easy to see that

$$\sum_{k=1}^{\infty} (|a_k| + |b_k|) \leq \sum_{k=1}^{\infty} [2(a_k^2 + b_k^2)]^{1/2}.$$

This indicates that the series $\sum_{k=1}^{\infty}(|a_k| + |b_k|)$ is convergent by the comparison test.

Note that, in general, we should not expect that a term-by-term differentiation of a Fourier series of $f(x)$ will result in a Fourier series of $f'(x)$. For example, for the function $f(x) = x$, $-\pi \leq x \leq \pi$, the Fourier series is (see Example 11.2.1)

$$x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx.$$

Differentiating this series term by term, we obtain $\sum_{n=1}^{\infty} 2(-1)^{n+1} \cos nx$. This, however, is not the Fourier series of $f'(x) = 1$, since the Fourier series of $f'(x) = 1$ is just 1. Note that in this case, $f(\pi) \neq f(-\pi)$, which violates one of the conditions in Theorem 11.3.1.

Theorem 11.3.2. If $f(x)$ is piecewise continuous on $[-\pi, \pi]$ and has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx], \quad (11.32)$$

then a term-by-term integration of this series gives the Fourier series of $\int_{-\pi}^x f(t) dt$ for $x \in [-\pi, \pi]$, that is,

$$\int_{-\pi}^x f(t) dt = \frac{a_0(\pi + x)}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{n} \sin nx - \frac{b_n}{n} (\cos nx - \cos n\pi) \right],$$

$-\pi \leq x \leq \pi.$

Furthermore, the integrated series converges uniformly to $\int_{-\pi}^x f(t) dt$.

Proof. Define the function $g(x)$ as

$$g(x) = \int_{-\pi}^x f(t) dt - \frac{a_0}{2}x. \quad (11.33)$$

If $f(x)$ is piecewise continuous on $[-\pi, \pi]$, then it is Riemann integrable there, and by Theorem 6.4.7, $g(x)$ is continuous on $[-\pi, \pi]$. Furthermore, by Theorem 6.4.8, at each point where $f(x)$ is continuous, $g(x)$ is differentiable and

$$g'(x) = f(x) - \frac{a_0}{2}. \quad (11.34)$$

This implies that $g'(x)$ is piecewise continuous on $[-\pi, \pi]$. In addition, $g(-\pi) = g(\pi)$. To show this, we have from (11.33)

$$\begin{aligned} g(-\pi) &= \int_{-\pi}^{-\pi} f(t) dt + \frac{a_0}{2} \pi \\ &= \frac{a_0}{2} \pi. \\ g(\pi) &= \int_{-\pi}^{\pi} f(t) dt - \frac{a_0}{2} \pi \\ &= a_0 \pi - \frac{a_0}{2} \pi \\ &= \frac{a_0}{2} \pi, \end{aligned}$$

by the definition of a_0 . Thus, the function $g(x)$ satisfies the conditions of Theorem 11.3.1. It follows that the Fourier series of $g(x)$ converges uniformly to $g(x)$ on $[-\pi, \pi]$. We therefore have

$$g(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos nx + B_n \sin nx]. \quad (11.35)$$

Moreover, by part (a) of Theorem 11.3.1, we have

$$g'(x) = \sum_{n=1}^{\infty} [nB_n \cos nx - nA_n \sin nx]. \quad (11.36)$$

Then, from (11.32), (11.34), and (11.36), we obtain

$$\begin{aligned} a_n &= nB_n, & n &= 1, 2, \dots, \\ b_n &= -nA_n, & n &= 1, 2, \dots. \end{aligned}$$

Substituting in (11.35), we get

$$g(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[-\frac{b_n}{n} \cos nx + \frac{a_n}{n} \sin nx \right].$$

From (11.33) we then have

$$\int_{-\pi}^x f(t) dt = \frac{a_0 x}{2} + \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[-\frac{b_n}{n} \cos nx + \frac{a_n}{n} \sin nx \right]. \quad (11.37)$$

To find the value of A_0 , we set $x = -\pi$ in (11.37), which gives

$$0 = -\frac{a_0\pi}{2} + \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(-\frac{b_n}{n} \cos n\pi \right).$$

Hence,

$$\frac{A_0}{2} = \frac{a_0\pi}{2} + \sum_{n=1}^{\infty} \frac{b_n}{n} \cos n\pi.$$

Substituting $A_0/2$ in (11.37), we finally obtain

$$\int_{-\pi}^x f(t) dt = \frac{a_0(\pi+x)}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{n} \sin nx - \frac{b_n}{n} (\cos nx - \cos n\pi) \right]. \quad \square$$

11.4. THE FOURIER INTEGRAL

We have so far considered Fourier series corresponding to a function defined on the interval $[-\pi, \pi]$. As was seen earlier in this chapter, if a function is initially defined on $[-\pi, \pi]$, we can extend its definition outside $[-\pi, \pi]$ by considering its periodic extension. For example, if $f(-\pi) = f(\pi)$, then we can define $f(x)$ everywhere in $(-\infty, \infty)$ by requiring that $f(x+2\pi) = f(x)$ for all x . The choice of the interval $[-\pi, \pi]$ was made mainly for convenience. More generally, we can now consider a function $f(x)$ defined on the interval $[-c, c]$. For such a function, the corresponding Fourier series is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right) \right], \quad (11.38)$$

where

$$a_n = \frac{1}{c} \int_{-c}^c f(x) \cos\left(\frac{n\pi x}{c}\right) dx, \quad n = 0, 1, 2, \dots, \quad (11.39)$$

$$b_n = \frac{1}{c} \int_{-c}^c f(x) \sin\left(\frac{n\pi x}{c}\right) dx, \quad n = 1, 2, \dots \quad (11.40)$$

Now, a question arises as to what to do when we have a function $f(x)$ that is already defined everywhere on $(-\infty, \infty)$, but is not periodic. We shall show that, under certain conditions, such a function can be represented by an infinite integral rather than by an infinite series. This integral is called a Fourier integral. We now show the development of such an integral.

Substituting the expressions for a_n and b_n given by (11.39), (11.40) into (11.38), we obtain the Fourier series

$$\frac{1}{2c} \int_{-c}^c f(t) dt + \frac{1}{c} \sum_{n=1}^{\infty} \int_{-c}^c f(t) \cos\left[\frac{n\pi}{c}(t-x)\right] dt. \quad (11.41)$$

If c is finite and $f(x)$ satisfies the conditions of Corollary 11.2.3 on $[-c, c]$, then the Fourier series (11.41) converges to $\frac{1}{2}[f(x^-) + f(x^+)]$. However, this series representation of $f(x)$ is not valid outside the interval $[-c, c]$ unless $f(x)$ is periodic with the period $2c$.

In order to provide a representation that is valid for all values of x when $f(x)$ is not periodic, we need to consider extending the series in (11.41) by letting c go to infinity, assuming that $f(x)$ is absolutely integrable over the whole real line. We now show how this can be done:

As $c \rightarrow \infty$, the first term in (11.41) goes to zero provided that $\int_{-\infty}^{\infty} f(t) dt$ exists. To investigate the limit of the series in (11.41) as $c \rightarrow \infty$, we set $\lambda_1 = \pi/c$, $\lambda_2 = 2\pi/c$, $\dots$, $\lambda_n = n\pi/c$, $\dots$, $\Delta\lambda_n = \lambda_{n+1} - \lambda_n = \pi/c$, $n = 1, 2, \dots$. We can then write

$$\frac{1}{c} \sum_{n=1}^{\infty} \int_{-c}^c f(t) \cos \left[\frac{n\pi}{c} (t-x) \right] dt = \frac{1}{\pi} \sum_{n=1}^{\infty} \Delta\lambda_n \int_{-c}^c f(t) \cos [\lambda_n (t-x)] dt. \quad (11.42)$$

When c is large, $\Delta\lambda_n$ is small, and the right-hand side of (11.42) will be an approximation of the integral

$$\frac{1}{\pi} \int_0^{\infty} \left\{ \int_{-\infty}^{\infty} f(t) \cos [\lambda(t-x)] dt \right\} d\lambda. \quad (11.43)$$

This is the Fourier integral of $f(x)$. Note that (11.43) can be written as

$$\int_0^{\infty} [a(\lambda) \cos \lambda x + b(\lambda) \sin \lambda x] d\lambda, \quad (11.44)$$

where

$$a(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \lambda t dt,$$

$$b(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \lambda t dt.$$

The expression in (11.44) resembles a Fourier series where the sum has been replaced by an integral and the parameter λ is used in place of the integer n . Moreover, $a(\lambda)$ and $b(\lambda)$ act like Fourier coefficients.

We now show that the Fourier integral in (11.43) provides a representation for $f(x)$ provided that $f(x)$ satisfies the conditions of the next theorem.

Theorem 11.4.1. Let $f(x)$ be piecewise continuous on every finite interval $[a, b]$. If $\int_{-\infty}^{\infty} |f(x)| dx$ exists, then at every point x ($-\infty < x < \infty$) where

$f'(x^+)$ and $f'(x^-)$ exist, the Fourier integral of $f(x)$ converges to $\frac{1}{2}[f(x^-) + f(x^+)]$, that is,

$$\frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^\infty f(t) \cos[\lambda(t-x)] dt \right\} d\lambda = \frac{1}{2}[f(x^-) + f(x^+)].$$

The proof of this theorem depends on the following lemmas:

Lemma 11.4.1. If $f(x)$ is piecewise continuous on $[a, b]$, then

$$\lim_{n \rightarrow \infty} \int_a^b f(x) \sin nx dx = 0, \quad (11.45)$$

$$\lim_{n \rightarrow \infty} \int_a^b f(x) \cos nx dx = 0. \quad (11.46)$$

Proof. Let the interval $[a, b]$ be divided into a finite number of subintervals on each of which $f(x)$ is continuous. Let any one of these subintervals be denoted by $[p, q]$. To prove formula (11.45) we only need to show that

$$\lim_{n \rightarrow \infty} \int_p^q f(x) \sin nx dx = 0. \quad (11.47)$$

For this purpose, we divide the interval $[p, q]$ into k equal subintervals using the partition points $x_0 = p, x_1, x_2, \dots, x_k = q$. We can then write the integral in (11.47) as

$$\sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} f(x) \sin nx dx,$$

or equivalently as

$$\sum_{i=0}^{k-1} \left\{ f(x_i) \int_{x_i}^{x_{i+1}} \sin nx dx + \int_{x_i}^{x_{i+1}} [f(x) - f(x_i)] \sin nx dx \right\}.$$

It follows that

$$\begin{aligned} \left| \int_p^q f(x) \sin nx dx \right| &\leq \sum_{i=0}^{k-1} \left| f(x_i) \frac{\cos nx_i - \cos nx_{i+1}}{n} \right| \\ &\quad + \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |f(x) - f(x_i)| dx. \end{aligned}$$

Let M denote the maximum value of $|f(x)|$ on $[p, q]$. Then

$$\left| \int_p^q f(x) \sin nx \, dx \right| \leq \frac{2Mk}{n} + \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |f(x) - f(x_i)| \, dx. \quad (11.48)$$

Furthermore, since $f(x)$ is continuous on $[p, q]$, it is uniformly continuous there [if necessary, $f(x)$ can be made continuous at p, q by simply using $f(p^+)$ and $f(q^-)$ as the values of $f(x)$ at p, q , respectively]. Hence, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \frac{\epsilon}{2(q-p)} \quad (11.49)$$

if $|x_1 - x_2| < \delta$, where x_1 and x_2 are points in $[p, q]$. If k is chosen large enough so that $|x_{i+1} - x_i| < \delta$, and hence $|x - x_i| < \delta$ if $x_i \leq x \leq x_{i+1}$, then from (11.48) we obtain

$$\left| \int_p^q f(x) \sin nx \, dx \right| \leq \frac{2Mk}{n} + \frac{\epsilon}{2(q-p)} \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} dx,$$

or

$$\left| \int_p^q f(x) \sin nx \, dx \right| \leq \frac{2Mk}{n} + \frac{\epsilon}{2},$$

since

$$\begin{aligned} \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} dx &= \sum_{i=0}^{k-1} (x_{i+1} - x_i) \\ &= q - p. \end{aligned}$$

Choosing n large enough so that $2Mk/n < \epsilon/2$, we finally get

$$\left| \int_p^q f(x) \sin nx \, dx \right| < \epsilon. \quad (11.50)$$

Formula (11.45) follows from (11.50), since $\epsilon > 0$ is arbitrary. Formula (11.46) can be proved in a similar fashion. $\square$

Lemma 11.4.2. If $f(x)$ is piecewise continuous on $[0, b]$ and $f'(0^+)$ exists, then

$$\lim_{n \rightarrow \infty} \int_0^b f(x) \frac{\sin nx}{x} \, dx = \frac{\pi}{2} f(0^+).$$

Proof. We have that

$$\begin{aligned} \int_0^b f(x) \frac{\sin nx}{x} dx &= f(0^+) \int_0^b \frac{\sin nx}{x} dx \\ &\quad + \int_0^b \frac{f(x) - f(0^+)}{x} \sin nx dx. \end{aligned} \quad (11.51)$$

But

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^b \frac{\sin nx}{x} dx &= \lim_{n \rightarrow \infty} \int_0^{bn} \frac{\sin x}{x} dx \\ &= \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \end{aligned}$$

(see Gillespie, 1959, page 89). Furthermore, the function $(1/x)[f(x) - f(0^+)]$ is piecewise continuous on $[0, b]$, since $f(x)$ is, and

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0^+)}{x} = f'(0^+),$$

which exists. Hence, by Lemma 11.4.1,

$$\lim_{n \rightarrow \infty} \int_0^b \frac{f(x) - f(0^+)}{x} \sin nx dx = 0.$$

From (11.51) we then have

$$\lim_{n \rightarrow \infty} \int_0^b f(x) \frac{\sin nx}{x} dx = \frac{\pi}{2} f(0^+). \quad \square$$

Lemma 11.4.3. If $f(x)$ is piecewise continuous on $[a, b]$, and $f'(x_0^-)$, $f'(x_0^+)$ exist at x_0 , $a < x_0 < b$, then

$$\lim_{n \rightarrow \infty} \int_a^b f(x) \frac{\sin[n(x - x_0)]}{x - x_0} dx = \frac{\pi}{2} [f(x_0^-) + f(x_0^+)].$$

Proof. We have that

$$\begin{aligned} \int_a^b f(x) \frac{\sin[n(x - x_0)]}{x - x_0} dx &= \int_a^{x_0} f(x) \frac{\sin[n(x - x_0)]}{x - x_0} dx \\ &\quad + \int_{x_0}^b f(x) \frac{\sin[n(x - x_0)]}{x - x_0} dx \\ &= \int_0^{x_0 - a} f(x_0 - x) \frac{\sin nx}{x} dx \\ &\quad + \int_0^{b - x_0} f(x_0 + x) \frac{\sin nx}{x} dx. \end{aligned}$$

Lemma 11.4.2 applies to each of the above integrals, since the right-hand derivatives of $f(x_0 - x)$ and $f(x_0 + x)$ at $x = 0$ are $-f'(x_0^-)$ and $f'(x_0^+)$, respectively, and both derivatives exist. Furthermore,

$$\lim_{x \rightarrow 0^+} f(x_0 - x) = f(x_0^-)$$

and

$$\lim_{x \rightarrow 0^+} f(x_0 + x) = f(x_0^+).$$

It follows that

$$\lim_{n \rightarrow \infty} \int_a^b f(x) \frac{\sin[n(x - x_0)]}{x - x_0} dx = \frac{\pi}{2} [f(x_0^-) + f(x_0^+)]. \quad \square$$

Proof of Theorem 11.4.1. The function $f(x)$ satisfies the conditions of Lemma 11.4.3 on the interval $[a, b]$. Hence, at any point x , $a < x < b$, where $f'(x_0^-)$ and $f'(x_0^+)$ exist,

$$\lim_{\lambda \rightarrow \infty} \int_a^b f(t) \frac{\sin[\lambda(t - x)]}{t - x} dt = \frac{\pi}{2} [f(x^-) + f(x^+)]. \quad (11.52)$$

Let us now partition the integral

$$I = \int_{-\infty}^{\infty} f(t) \frac{\sin[\lambda(t - x)]}{t - x} dt$$

as

$$\begin{aligned} I &= \int_{-\infty}^a f(t) \frac{\sin[\lambda(t - x)]}{t - x} dt + \int_a^b f(t) \frac{\sin[\lambda(t - x)]}{t - x} dt \\ &\quad + \int_b^{\infty} f(t) \frac{\sin[\lambda(t - x)]}{t - x} dt. \end{aligned} \quad (11.53)$$

From the first integral in (11.53) we have

$$\left| \int_{-\infty}^a f(t) \frac{\sin[\lambda(t - x)]}{t - x} dt \right| \leq \int_{-\infty}^a \frac{|f(t)|}{|t - x|} dt.$$

Since $t \leq a$ and $a < x$, then $|t - x| \geq x - a$. Hence,

$$\int_{-\infty}^a \frac{|f(t)|}{|t - x|} dt \leq \frac{1}{x - a} \int_{-\infty}^a |f(t)| dt. \quad (11.54)$$

The integral on the right-hand side of (11.54) exists because $\int_{-\infty}^{\infty} |f(t)| dt$ does. Similarly, from the third integral in (11.53) we have, if $x < b$,

$$\begin{aligned} \left| \int_b^{\infty} f(t) \frac{\sin[\lambda(t-x)]}{t-x} dt \right| &\leq \int_b^{\infty} \frac{|f(t)|}{|t-x|} dt \\ &\leq \frac{1}{b-x} \int_b^{\infty} |f(t)| dt \\ &\leq \frac{1}{b-x} \int_{-\infty}^{\infty} |f(t)| dt. \end{aligned}$$

Hence, the first and third integrals in (11.53) are convergent. It follows that for any $\epsilon > 0$, there exists a positive number N such that if $a < -N$ and $b > N$, then these integrals will each be less than $\epsilon/3$ in absolute value. Furthermore, by (11.52), the absolute value of the difference between the second integral in (11.53) and the value $(\pi/2)[f(x^-) + f(x^+)]$ can be made less than $\epsilon/3$, if λ is chosen large enough. Consequently, the absolute value of the difference between the value of the integral I and $(\pi/2)[f(x^-) + f(x^+)]$ will be less than ϵ , if λ is chosen large enough. Thus,

$$\lim_{\lambda \rightarrow \infty} \int_{-\infty}^{\infty} f(t) \frac{\sin[\lambda(t-x)]}{t-x} dt = \frac{\pi}{2} [f(x^-) + f(x^+)]. \quad (11.55)$$

The expression $\sin[\lambda(t-x)]/(t-x)$ in (11.55) can be written as

$$\frac{\sin[\lambda(t-x)]}{t-x} = \int_0^{\lambda} \cos[\alpha(t-x)] d\alpha.$$

Formula (11.55) can then be expressed as

$$\begin{aligned} \frac{1}{2} [f(x^-) + f(x^+)] &= \frac{1}{\pi} \lim_{\lambda \rightarrow \infty} \int_{-\infty}^{\infty} f(t) dt \int_0^{\lambda} \cos[\alpha(t-x)] d\alpha \\ &= \frac{1}{\pi} \lim_{\lambda \rightarrow \infty} \int_0^{\lambda} d\alpha \int_{-\infty}^{\infty} f(t) \cos[\alpha(t-x)] dt. \end{aligned} \quad (11.56)$$

The change of the order of integration in (11.56) is valid because the integrand in (11.56) does not exceed $|f(t)|$ in absolute value, so that the integral $\int_{-\infty}^{\infty} f(t) \cos[\alpha(t-x)] dt$ converges uniformly for all α (see Carslaw, 1930, page 199; Pinkus and Zafrany, 1997, page 187). From (11.56) we finally obtain

$$\frac{1}{2} [f(x^-) + f(x^+)] = \frac{1}{\pi} \int_0^{\infty} \left\{ \int_{-\infty}^{\infty} f(t) \cos[\alpha(t-x)] dt \right\} d\alpha. \quad \square$$

11.5. APPROXIMATION OF FUNCTIONS BY TRIGONOMETRIC POLYNOMIALS

By a trigonometric polynomial of the n th order it is meant an expression of the form

$$t_n(x) = \frac{\alpha_0}{2} + \sum_{k=1}^n [\alpha_k \cos kx + \beta_k \sin kx]. \quad (11.57)$$

A theorem of Weierstrass states that any continuous function of period 2π can be uniformly approximated by a trigonometric polynomial of some order (see, for example, Tolstov, 1962, Chapter 5). Thus, for a given $\epsilon > 0$, there exists a trigonometric polynomial of the form (11.57) such that

$$|f(x) - t_n(x)| < \epsilon$$

for all values of x . In case the Fourier series for $f(x)$ is uniformly convergent, then $t_n(x)$ can be chosen to be equal to $s_n(x)$, the n th partial sum of the Fourier series. However, it should be noted that $t_n(x)$ is not merely a partial sum of the Fourier series for $f(x)$, since a continuous function may have a divergent Fourier series (see Jackson, 1941, page 26). We now show that $s_n(x)$ has a certain optimal property among all trigonometric polynomials of the same order. To demonstrate this fact, let $f(x)$ be Riemann integrable on $[-\pi, \pi]$, and let $s_n(x)$ be the partial sum of order n of its Fourier series, that is, $s_n(x) = a_0/2 + \sum_{k=1}^n [a_k \cos kx + b_k \sin kx]$. Let $r_n(x) = f(x) - s_n(x)$. Then, from (11.2),

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos kx dx &= \int_{-\pi}^{\pi} s_n(x) \cos kx dx \\ &= \pi a_k, \quad k = 0, 1, \dots, n. \end{aligned}$$

Hence,

$$\int_{-\pi}^{\pi} r_n(x) \cos kx dx = 0 \quad \text{for } k \leq n. \quad (11.58)$$

We can similarly show that

$$\int_{-\pi}^{\pi} r_n(x) \sin kx dx = 0 \quad \text{for } k \leq n. \quad (11.59)$$

Now, let $u_n(x) = t_n(x) - s_n(x)$, where $t_n(x)$ is given by (11.57). Then

$$\begin{aligned} \int_{-\pi}^{\pi} [f(x) - t_n(x)]^2 dx &= \int_{-\pi}^{\pi} [r_n(x) - u_n(x)]^2 dx \\ &= \int_{-\pi}^{\pi} r_n^2(x) dx - 2 \int_{-\pi}^{\pi} r_n(x) u_n(x) dx + \int_{-\pi}^{\pi} u_n^2(x) dx \\ &= \int_{-\pi}^{\pi} [f(x) - s_n(x)]^2 dx + \int_{-\pi}^{\pi} u_n^2(x) dx, \quad (11.60) \end{aligned}$$

since, by (11.58) and (11.59),

$$\int_{-\pi}^{\pi} r_n(x) u_n(x) dx = 0.$$

From (11.60) it follows that

$$\int_{-\pi}^{\pi} [f(x) - t_n(x)]^2 dx \geq \int_{-\pi}^{\pi} [f(x) - s_n(x)]^2 dx. \quad (11.61)$$

This shows that for all trigonometric polynomials of order n , $\int_{-\pi}^{\pi} [f(x) - t_n(x)]^2 dx$ is minimized when $t_n(x) = s_n(x)$.

11.5.1. Parseval's Theorem

Suppose that we have the Fourier series (11.5) for the function $f(x)$, which is assumed to be continuous of period 2π . Let $s_n(x)$ be the n th partial sum of the series. We recall from the proof of Lemma 11.2.1 that

$$\int_{-\pi}^{\pi} [f(x) - s_n(x)]^2 dx = \int_{-\pi}^{\pi} f^2(x) dx - \left[\frac{\pi a_0^2}{2} + \pi \sum_{k=1}^n (a_k^2 + b_k^2) \right]. \quad (11.62)$$

We also recall that for a given $\epsilon > 0$, there exists a trigonometric polynomial $t_n(x)$ of order n such that

$$|f(x) - t_n(x)| < \epsilon.$$

Hence,

$$\int_{-\pi}^{\pi} [f(x) - t_n(x)]^2 dx < 2\pi\epsilon^2.$$

Applying (11.61), we obtain

$$\begin{aligned} \int_{-\pi}^{\pi} [f(x) - s_n(x)]^2 dx &\leq \int_{-\pi}^{\pi} [f(x) - t_n(x)]^2 dx \\ &< 2\pi\epsilon^2. \end{aligned} \quad (11.63)$$

Since $\epsilon > 0$ is arbitrary, we may conclude from (11.62) and (11.63) that the limit of the right-hand side of (11.62) is zero as $n \rightarrow \infty$, that is,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

This result is known as *Parseval's theorem* after Marc Antoine Parseval (1755–1836).

11.6. THE FOURIER TRANSFORM

In the previous sections we discussed Fourier series for functions defined on a finite interval (or periodic functions defined on R , the set of all real numbers). In this section, we study a particular transformation of functions defined on R which are not periodic.

Let $f(x)$ be defined on $R = (-\infty, \infty)$. The Fourier transform of $f(x)$ is a function defined on R as

$$F(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx, \quad (11.64)$$

where i the complex number $\sqrt{-1}$, and

$$e^{-iwx} = \cos wx - i \sin wx.$$

A proper understanding of such a transformation requires some knowledge of complex analysis, which is beyond the scope of this book. However, due to the importance and prevalence of the use of this transformation in various fields of science and engineering, some coverage of its properties is necessary. For this reason, we merely state some basic results and properties concerning this transformation. For more details, the reader is referred to standard books on Fourier series, for example, Pinkus and Zafrany (1997, Chapter 3), Kufner and Kadlec (1971, Chapter 8), and Weaver (1989, Chapter 6).

Theorem 11.6.1. If $f(x)$ is absolutely integrable on R , then its Fourier transform $F(w)$ exists.

Theorem 11.6.2. If $f(x)$ is piecewise continuous and absolutely integrable on R , then its Fourier transform $F(w)$ has the following properties:

- a. $F(w)$ is a continuous function on R .
- b. $\lim_{w \rightarrow \mp \infty} F(w) = 0$.

Note that $f(x)$ is piecewise continuous on R if it is piecewise continuous on each finite interval $[a, b]$.

EXAMPLE 11.6.1. Let $f(x) = e^{-|x|}$. This function is absolutely integrable on R , since

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-|x|} dx &= 2 \int_0^{\infty} e^{-x} dx \\ &= 2. \end{aligned}$$

Its Fourier transform is given by

$$\begin{aligned} F(w) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|x|} e^{-iwx} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|x|} (\cos wx - i \sin wx) dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|x|} \cos wx dx \\ &= \frac{1}{\pi} \int_0^{\infty} e^{-x} \cos wx dx. \end{aligned}$$

Integrating by parts twice, it can be shown that

$$F(w) = \frac{1}{\pi(1+w^2)}.$$

EXAMPLE 11.6.2. Consider the function

$$f(x) = \begin{cases} 1 & |x| \leq a, \\ 0 & \text{otherwise,} \end{cases}$$

where a is a finite positive number. This function is absolutely integrable on R , since

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)| dx &= \int_{-a}^a dx \\ &= 2a. \end{aligned}$$

Its Fourier transform is given by

$$\begin{aligned} F(w) &= \frac{1}{2\pi} \int_{-a}^a e^{-iwx} dx \\ &= \frac{1}{2\pi iw} (e^{iwa} - e^{-iwa}) \\ &= \frac{\sin wa}{\pi w}. \end{aligned}$$

The next theorem gives the condition that makes it possible to express the function $f(x)$ in terms of its Fourier transform using the so-called *inverse Fourier transform*.

Theorem 11.6.3. Let $f(x)$ be piecewise continuous and absolutely integrable on R . Then for every point $x \in R$ where $f'(x^-)$ and $f'(x^+)$ exist, we have

$$\frac{1}{2} [f(x^-) + f(x^+)] = \int_{-\infty}^{\infty} F(w) e^{iwx} dw.$$

In particular, if $f(x)$ is continuous on R , then

$$f(x) = \int_{-\infty}^{\infty} F(w) e^{iwx} dw. \quad (11.65)$$

By applying Theorem 11.6.3 to the function in Example 11.6.1, we obtain

$$\begin{aligned} e^{-|x|} &= \int_{-\infty}^{\infty} \frac{e^{iwx}}{\pi(1+w^2)} dw \\ &= \int_{-\infty}^{\infty} \frac{1}{\pi(1+w^2)} (\cos wx + i \sin wx) dw \\ &= \int_{-\infty}^{\infty} \frac{\cos wx}{\pi(1+w^2)} dw \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{\cos wx}{1+w^2} dw. \end{aligned}$$

11.6.1. Fourier Transform of a Convolution

Let $f(x)$ and $g(x)$ be absolutely integrable functions on R . By definition, the function

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y) dy \quad (11.66)$$

is called the convolution of $f(x)$ and $g(x)$ and is denoted by $(f * g)(x)$.

Theorem 11.6.4. Let $f(x)$ and $g(x)$ be absolutely integrable on R . Let $F(w)$ and $G(w)$ be their respective Fourier transforms. Then, the Fourier transform of the convolution $(f * g)(x)$ is given by $2\pi F(w)G(w)$.

11.7. APPLICATIONS IN STATISTICS

Fourier series have been used in a wide variety of areas in statistics, such as time series, stochastic processes, approximation of probability distribution functions, and the modeling of a periodic response variable, to name just a few. In addition, the methods and results of Fourier analysis have been effectively utilized in the analytic theory of probability (see, for example, Kawata, 1972).

11.7.1. Applications in Time Series

A time series is a collection of observations made sequentially in time. Examples of time series can be found in a variety of fields ranging from economics to engineering. Many types of time series occur in the physical sciences, particularly in meteorology, such as the study of rainfall on successive days, as well as in marine science and geophysics.

The stimulus for the use of Fourier methods in time series analysis is the recognition that when observing data over time, some aspects of an observed physical phenomenon tend to exhibit cycles or periodicities. Therefore, when considering a model to represent such data, it is natural to use models that contain sines and cosines, that is, trigonometric models, to describe the behavior. Let $y_1, y_2, \dots, y_n$ denote a time series consisting of n observations obtained over time. These observations can be represented by the trigonometric polynomial model

$$y_t = \frac{a_0}{2} + \sum_{k=1}^m [a_k \cos \omega_k t + b_k \sin \omega_k t], \quad t = 1, 2, \dots, n,$$

where

$$\omega_k = \frac{2\pi k}{n}, \quad k = 0, 1, 2, \dots, m,$$

$$a_k = \frac{2}{n} \sum_{t=1}^n y_t \cos \omega_k t, \quad k = 0, 1, \dots, m,$$

$$b_k = \frac{2}{n} \sum_{t=1}^n y_t \sin \omega_k t, \quad k = 1, 2, \dots, m.$$

The values $\omega_1, \omega_2, \dots, \omega_m$ are called harmonic frequencies. This model provides a decomposition of the time series into a set of cycles based on the harmonic frequencies. Here, n is assumed to be odd and equal to $2m + 1$, so that the harmonic frequencies lie in the range 0 to π . The expressions for a_k ($k = 0, 1, \dots, m$) and b_k ($k = 1, 2, \dots, m$) were obtained by treating the model as a linear regression model with $2m + 1$ parameters and then fitting it to the $2m + 1$ observations by the method of least squares. See, for example, Fuller (1976, Chapter 7).

The quantity

$$I_n(\omega_k) = \frac{n}{2}(a_k^2 + b_k^2), \quad k = 1, 2, \dots, m, \quad (11.67)$$

represents the sum of squares associated with the frequency ω_k . For $k = 1, 2, \dots, m$, the quantities in (11.67) define the so-called *periodogram*.

If $y_1, y_2, \dots, y_n$ are independently distributed as normal variates with zero means and variances σ^2 , then the a_k 's and b_k 's, being linear combinations of the y_i 's, will be normally distributed. They are also independent, since the sine and cosine functions are orthogonal. It follows that $[n/(2\sigma^2)](a_k^2 + b_k^2)$, for $k = 1, 2, \dots, m$, are distributed as independent chi-squared variates with two degrees of freedom each. The periodogram can be used to search for cycles or periodicities in the data.

Much of time series data analysis is based on the Fourier transform and its efficient computation. For more details concerning Fourier analysis of time series, the reader is referred to Bloomfield (1976) and Otnes and Enochson (1978).

11.7.2. Representation of Probability Distributions

One of the interesting applications of Fourier series in statistics is in providing a representation that can be used to evaluate the distribution function of a random variable with a finite range. Woods and Posten (1977) introduced two such representations by combining the concepts of Fourier series and Chebyshev polynomials of the first kind (see Section 10.4.1). These representations are given by the following two theorems:

Theorem 11.7.1. Let X be a random variable with a cumulative distribution function $F(x)$ defined on $[0, 1]$. Then, $F(x)$ can be represented as a Fourier series of the form

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - \theta/\pi - \sum_{n=1}^{\infty} b_n \sin n\theta, & 0 \leq x \leq 1, \\ 1, & x > 1, \end{cases}$$

where $\theta = \text{Arccos}(2x - 1)$, $b_n = [2/(n\pi)]E[T_n^*(X)]$, and $E[T_n^*(X)]$ is the expected value of the random variable

$$T_n^*(X) = \cos[n \text{Arccos}(2X - 1)], \quad 0 \leq X \leq 1. \quad (11.68)$$

Note that $T_n^*(x)$ is basically a Chebyshev polynomial of the first kind and of the n th degree defined on $[0, 1]$.

Proof. See Theorem 1 in Woods and Posten (1977). $\square$

The second representation theorem is similar to Theorem 11.7.1, except that X is now assumed to be a random variable over $[-1, 1]$.

Theorem 11.7.2. Let X be a random variable with a cumulative distribution function $F(x)$ defined on $[-1, 1]$. Then

$$F(x) = \begin{cases} 0, & x < -1, \\ 1 - \theta/\pi - \sum_{n=1}^{\infty} b_n \sin n\theta, & -1 \leq x \leq 1, \\ 1, & x > 1, \end{cases}$$

where $\theta = \text{Arccos } x$, $b_n = [2/(n\pi)]E[T_n(X)]$, $E[T_n(X)]$ is the expected value of the random variable

$$T_n(X) = \cos[n \text{Arccos } X], \quad -1 \leq X \leq 1,$$

and $T_n(x)$ is Chebyshev polynomial of the first kind and the n th degree [see formula (10.12)].

Proof. See Theorem 2 in Woods and Posten (1977). $\square$

To evaluate the Fourier series representation of $F(x)$, we must first compute the coefficients b_n . For example, in Theorem 11.7.2, $b_n = [2/(n\pi)]E[T_n(X)]$. Since the Chebyshev polynomial $T_n(x)$ can be written in the form

$$T_n(x) = \sum_{k=0}^n \alpha_{nk} x^k, \quad n = 1, 2, \dots,$$

the computation of b_n is equivalent to evaluating

$$b_n = \frac{2}{n\pi} \sum_{k=0}^n \alpha_{nk} \mu'_k, \quad n = 1, 2, \dots,$$

where $\mu'_k = E(X^k)$ is the k th noncentral moment of X . The coefficients α_{nk} can be obtained by using the recurrence relation (10.16), that is,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, \dots,$$

with $T_0(x) = 1$, $T_1(x) = x$. This allows us to evaluate the α_{nk} 's recursively. The series

$$F(x) = 1 - \frac{\theta}{\pi} - \sum_{n=1}^{\infty} b_n \sin n\theta$$

is then truncated at $n = N$. Thus

$$F(x) \approx 1 - \frac{\theta}{\pi} - \sum_{k=1}^N b_k \sin k\theta.$$

Several values of N can be tried to determine the sensitivity of the approximation. We note that this series expansion provides an approximation of $F(x)$ in terms of the noncentral moments of X . Good estimates of these moments should therefore be available.

It is also possible to extend the applications of Theorems 11.7.1 and 11.7.2 to a random variable X with an infinite range provided that there exists a transformation which transforms X to a random variable Y over $[0, 1]$, or over $[-1, 1]$, such that the moments of Y are known from the moments of X .

In another application, Fettis (1976) developed a Fourier series expansion for Pearson Type IV distributions. These are density functions, $f(x)$, that satisfy the differential equation

$$\frac{df(x)}{dx} = \frac{-(x+a)}{c_0 + c_1x + c_2x^2} f(x),$$

where a, c_0, c_1 , and c_2 are constants determined from the central moments μ_1, μ_2, μ_3 , and μ_4 , which can be estimated from the raw data. The data are standardized so that $\mu_1 = 0$, $\mu_2 = 1$. This results in the following expressions for a, c_0, c_1 , and c_2 :

$$\begin{aligned} c_0 &= \frac{2\alpha - 1}{2(\alpha + 1)}, \\ c_1 &= a \\ &= \frac{\mu_3(\alpha - 1)}{\alpha + 1}, \\ c_2 &= \frac{1}{2(\alpha + 1)}, \end{aligned}$$

where

$$\alpha = \frac{3(\mu_4 - \mu_3^2 - 1)}{2\mu_4 - 3\mu_3^2 - 6}.$$

Fettis (1976) provided additional details that explain how to approximate the cumulative distribution function,

$$F(x) = \int_{-\infty}^x f(t) dt,$$

using Fourier series.

11.7.3. Regression Modeling

In regression analysis and response surface methodology, it is quite common to use polynomial models to approximate the mean η of a response variable. There are, however, situations in which polynomial models are not adequate representatives of the mean response, as when η is known to be a periodic function. In this case, it is more appropriate to use an approximating function which is itself periodic.

Kupper (1972) proposed using partial sums of Fourier series as possible models to approximate the mean response. Consider the following trigonometric polynomial of order d ,

$$\eta = \alpha_0 + \sum_{u=1}^d [\alpha_n \cos n\phi + \beta_n \sin n\phi], \quad (11.69)$$

where $0 \leq \phi \leq 2\pi$ represents either a variable taking values on the real line between 0 and 2π , or the angle associated with the polar coordinates of a point on the unit circle. Let $\mathbf{u} = (u_1, u_2)'$, where $u_1 = \cos \phi$, $u_2 = \sin \phi$. Then, when $d = 2$, the model in (11.69) can be written as

$$\eta = \alpha_0 + \alpha_1 u_1 + \beta_1 u_2 + \alpha_2 u_1^2 - \alpha_2 u_2^2 + 2\beta_2 u_1 u_2, \quad (11.70)$$

since $\sin 2\phi = 2 \sin \phi \cos \phi = 2u_1 u_2$, and $\cos 2\phi = \cos^2 \phi - \sin^2 \phi = u_1^2 - u_2^2$.

One of the objectives of response surface methodology is the determination of optimum settings of the model's control variables that result in a maximum (or minimum) predicted response. The predicted response $\hat{y}$ at a point provides an estimate of η in (11.69) and is obtained by replacing $\alpha_0, \alpha_n, \beta_n$ in (11.69) by their least-squares estimates $\hat{\alpha}_0, \hat{\alpha}_n$, and $\hat{\beta}_n$, respectively, $n = 1, 2, \dots, d$. For example, if $d = 2$, we have

$$\hat{y} = \hat{\alpha}_0 + \sum_{n=1}^2 [\hat{\alpha}_n \cos n\phi + \hat{\beta}_n \sin n\phi], \quad (11.71)$$

which can be expressed using (11.70) as

$$\hat{y} = \hat{\alpha}_0 + \mathbf{u}'\hat{\mathbf{b}} + \mathbf{u}'\hat{\mathbf{B}}\mathbf{u},$$

where $\hat{\mathbf{b}} = (\hat{\alpha}_1, \hat{\beta}_1)'$ and

$$\hat{\mathbf{B}} = \begin{bmatrix} \hat{\alpha}_2 & \hat{\beta}_2 \\ \hat{\beta}_2 & -\hat{\alpha}_2 \end{bmatrix}$$

with $\mathbf{u}'\mathbf{u} = 1$. The method of Lagrange multipliers can then be used to determine the stationary points of $\hat{y}$ subject to the constraint $\mathbf{u}'\mathbf{u} = 1$. Details of this procedure are given in Kupper (1972, Section 3). In a follow-up paper, Kupper (1973) presented some results on the construction of optimal designs for model (11.69).

More recently, Anderson-Cook (2000) used model (11.71) in experimental situations involving cylindrical data. For such data, it is of interest to model the relationship between two correlated components, one a standard linear measurement y , and the other an angular measure ϕ . Examples of such data arise, for example, in biology (plant or animal migration patterns), and geology (direction and magnitude of magnetic fields). The fitting of model (11.71) is done by using the method of ordinary least squares with the assumption that y is normally distributed and has a constant variance. Anderson-Cook used an example, originally presented in Mardia and Sutton (1978), of a cylindrical data set in which y is temperature (measured in degrees Fahrenheit) and ϕ is wind direction (measured in radians). Based on this example, the fitted model is

$$\hat{y} = 41.33 - 2.43 \cos \phi - 2.60 \sin \phi + 3.05 \cos 2\phi + 2.98 \sin 2\phi.$$

The corresponding standard errors of $\hat{\alpha}_0$, $\hat{\alpha}_1$, $\hat{\beta}_1$, $\hat{\alpha}_2$, $\hat{\beta}_2$ are 1.1896, 1.6608, 1.7057, 1.4029, 1.7172, respectively. Both α_0 and α_2 are significant parameters at the 5% level, and β_2 is significant at the 10% level.

11.7.4. The Characteristic Function

We have seen that the moment generating function $\phi(t)$ for a random variable X is used to obtain the moments of X (see Section 5.6.2 and Example 6.9.8). It may be recalled, however, that $\phi(t)$ may not be defined for all values of t . To generate all the moments of X , it is sufficient for $\phi(t)$ to be defined in a neighborhood of $t = 0$ (see Section 5.6.2). Some well-known distributions do not have moment generating functions, such as the Cauchy distribution (see Example 6.9.1).

Another function that generates the moments of a random variable in a manner similar to $\phi(t)$, but is defined for all values of t and for all random

variables, is the *characteristic function*. By definition, the characteristic function of a random variable X , denoted by $\phi_c(t)$, is

$$\begin{aligned}\phi_c(t) &= E[e^{itX}] \\ &= \int_{-\infty}^{\infty} e^{itx} dF(x),\end{aligned}\tag{11.72}$$

where $F(x)$ is the cumulative distribution function of X , and i is the complex number $\sqrt{-1}$. If X is discrete and has the values $c_1, c_2, \dots, c_n, \dots$, then (11.72) takes the form

$$\phi_c(t) = \sum_{j=1}^{\infty} p(c_j) e^{itc_j},\tag{11.73}$$

where $p(c_j) = P[X = c_j]$, $j = 1, 2, \dots$. If X is continuous with the density function $f(x)$, then

$$\phi_c(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx.\tag{11.74}$$

The function $\phi_c(t)$ is complex-valued in general, but is defined for all values of t , since $e^{itx} = \cos tx + i \sin tx$, and both $\int_{-\infty}^{\infty} \cos tx dF(x)$ and $\int_{-\infty}^{\infty} \sin tx dF(x)$ exist by the fact that

$$\begin{aligned}\int_{-\infty}^{\infty} |\cos tx| dF(x) &\leq \int_{-\infty}^{\infty} dF(x) = 1, \\ \int_{-\infty}^{\infty} |\sin tx| dF(x) &\leq \int_{-\infty}^{\infty} dF(x) = 1.\end{aligned}$$

The characteristic function and the moment generating function, when the latter exists, are related according to the formula

$$\phi_c(t) = \phi(it).$$

Furthermore, it can be shown that if X has finite moments, then they can be obtained by repeatedly differentiating $\phi_c(t)$ and evaluating the derivatives at zero, that is,

$$E(X^n) = \frac{1}{i^n} \left. \frac{d^n \phi_c(t)}{dt^n} \right|_{t=0}, \quad n = 1, 2, \dots$$

Although $\phi_c(t)$ generates moments, it is mainly used as a tool to derive distributions. For example, from (11.74) we note that when X is continuous, the characteristic function is a Fourier-type transformation of the density

function $f(x)$. This follows from (11.64), the Fourier transform of $f(x)$, which is given by $(1/2\pi)\int_{-\infty}^{\infty} f(x)e^{-itx} dx$. If we denote this transform by $G(t)$, then the relationship between $\phi_c(t)$ and $G(t)$ is given by

$$\phi_c(t) = 2\pi G(-t).$$

By Theorem 11.6.3, if $f(x)$ is continuous and absolutely integrable on R , then $f(x)$ can be derived from $\phi_c(t)$ by using formula (11.65), which can be written as

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} G(t)e^{itx} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_c(-t)e^{itx} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_c(t)e^{-itx} dt. \end{aligned} \quad (11.75)$$

This is known as the *inversion formula* for characteristic functions. Thus the distribution of X can be uniquely determined by its characteristic function. There is therefore a one-to-one correspondence between distribution functions and their corresponding characteristic functions. This provides a useful tool for deriving distributions of random variables that cannot be easily calculated, but whose characteristic functions are straightforward. Waller, Turnbull, and Hardin (1995) reviewed and discussed several algorithms for inverting characteristic functions, and gave several examples from various areas in statistics. Waller (1995) demonstrated that characteristic functions provide information beyond what is given by moment generating functions. He pointed out that moment generating functions may be of more mathematical than numerical use in characterizing distributions. He used an example to illustrate that numerical techniques using characteristic functions can differentiate between two distributions, even though their moment generating functions are very similar (see also McCullagh, 1994).

Luceño (1997) provided further and more general arguments to show that characteristic functions are superior to moment generating and probability generating functions (see Section 5.6.2) in their numerical behavior.

One of the principal uses of characteristic functions is in deriving limiting distributions. This is based on the following theorem (see, for example, Pfeiffer, 1990, page 426):

Theorem 11.7.3. Consider the sequence $\{F_n(x)\}_{n=1}^{\infty}$ of cumulative distribution functions. Let $\{\phi_{nc}(t)\}_{n=1}^{\infty}$ be the corresponding sequence of characteristic functions.

- a. If $F_n(x)$ converges to a distribution function $F(x)$ at every point of continuity for $F(x)$, then $\phi_{nc}(t)$ converges to $\phi_c(t)$ for all t , where $\phi_c(t)$ is the characteristic function for $F(x)$.

- b. If $\phi_{nc}(t)$ converges to $\phi_c(t)$ for all t and $\phi_c(t)$ is continuous at $t=0$, then $\phi_c(t)$ is the characteristic function for a distribution function $F(x)$ such that $F_n(x)$ converges to $F(x)$ at each point of continuity of $F(x)$.

It should be noted that in Theorem 11.7.3 the condition that the limiting function $\phi_c(t)$ is continuous at $t=0$ is essential for the validity of the theorem. The following example shows that if this condition is violated, then the theorem is no longer true:

Consider the cumulative distribution function

$$F_n(x) = \begin{cases} 0, & x \leq -n, \\ \frac{x+n}{2n}, & -n < x < n, \\ 1, & x \geq n. \end{cases}$$

The corresponding characteristic function is

$$\begin{aligned} \phi_{nc}(t) &= \frac{1}{2n} \int_{-n}^n e^{itx} dx \\ &= \frac{\sin nt}{nt}. \end{aligned}$$

As $n \rightarrow \infty$, $\phi_{nc}(t)$ converges for every t to $\phi_c(t)$ defined by

$$\phi_c(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

Thus, $\phi_c(t)$ is not continuous for $t=0$. We note, however, that $F_n(x) \rightarrow \frac{1}{2}$ for every fixed x . Hence, the limit of $F_n(x)$ is not a cumulative distribution function.

EXAMPLE 11.7.1. Consider the distribution defined by the density function $f(x) = e^{-x}$ for $x > 0$. Its characteristic function is given by

$$\begin{aligned} \phi_c(t) &= \int_0^\infty e^{itx} e^{-x} dx \\ &= \int_0^\infty e^{-x(1-it)} dx \\ &= \frac{1}{1-it}. \end{aligned}$$

EXAMPLE 11.7.2. Consider the Cauchy density function

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty,$$

given in Example 6.9.1. The characteristic function is

$$\begin{aligned}
 \phi_c(t) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx \\
 &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos tx}{1+x^2} dx + \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin tx}{1+x^2} dx \\
 &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos tx}{1+x^2} dx \\
 &= e^{-|t|}.
 \end{aligned}$$

Note that this function is not differentiable at $t=0$. We may recall from Example 6.9.1 that all moments of the Cauchy distribution do not exist.

EXAMPLE 11.7.3. In Example 6.9.8 we saw that the moment generating function for the gamma distribution $G(\alpha, \beta)$ with the density function

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha) \beta^\alpha}, \quad \alpha > 0, \quad \beta > 0, \quad 0 < x < \infty,$$

is $\phi(t) = (1 - \beta t)^{-\alpha}$. Hence, its characteristic function is $\phi_c(t) = (1 - i\beta t)^{-\alpha}$.

EXAMPLE 11.7.4. The characteristic function of the standard normal distribution with the density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty,$$

is

$$\begin{aligned}
 \phi_c(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{itx} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2itx)} dx \\
 &= \frac{e^{-t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-it)^2} dx \\
 &= e^{-t^2/2}.
 \end{aligned}$$

Vice versa, the density function can be retrieved from $\phi_c(t)$ by using the inversion formula (11.75):

$$\begin{aligned}
 f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t^2/2} e^{-itx} dt \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t^2 + 2itx)} dt \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[t^2 + 2t(ix) + (ix)^2]} e^{(ix)^2/2} dt \\
 &= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t+ix)^2} dt \\
 &= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \\
 &= \frac{e^{-x^2/2}}{\sqrt{2\pi}}.
 \end{aligned}$$

11.7.4.1. Some Properties of Characteristic Functions

The book by Lukacs (1970) provides a detailed study of characteristic functions and their properties. Proofs of the following theorems can be found in Chapter 2 of that book.

Theorem 11.7.4. Every characteristic function is uniformly continuous on the whole real line.

Theorem 11.7.5. Suppose that $\phi_{1c}(t), \phi_{2c}(t), \dots, \phi_{nc}(t)$ are characteristic functions. Let $a_1, a_2, \dots, a_n$ be nonnegative numbers such that $\sum_{i=1}^n a_i = 1$. Then $\sum_{i=1}^n a_i \phi_{ic}(t)$ is also a characteristic function.

Theorem 11.7.6. The characteristic function of the convolution of two distribution functions is the product of their characteristic functions.

Theorem 11.7.7. The product of two characteristic functions is a characteristic function.

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EXERCISES

In Mathematics

11.1. Expand the following functions using Fourier series:

- (a) $f(x) = |x|$, $-\pi \leq x \leq \pi$.
 (b) $f(x) = |\sin x|$.
 (c) $f(x) = x + x^2$, $-\pi \leq x \leq \pi$.

11.2. Show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

[Hint: Use the Fourier series for x^2 .]

11.3. Let a_n and b_n be the Fourier coefficients for a continuous function $f(x)$ defined on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$, and $f'(x)$ is piecewise continuous on $[-\pi, \pi]$. Show that

- (a) $\lim_{n \rightarrow \infty} (na_n) = 0$,
 (b) $\lim_{n \rightarrow \infty} (nb_n) = 0$.

11.4. If $f(x)$ is continuous on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, and $f'(x)$ is piecewise continuous on $[-\pi, \pi]$, then show that

$$|f(x) - s_n(x)| \leq \frac{c}{\sqrt{n}},$$

where

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^n [a_k \cos kx + b_k \sin kx],$$

and

$$c^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f'^2(x) dx.$$

11.5. Suppose that $f(x)$ is piecewise continuous on $[-\pi, \pi]$ and has the Fourier series given in (11.32).

(a) Show that $\sum_{n=1}^{\infty} (-1)^n b_n/n$ is a convergent series.

(b) Show that $\sum_{n=1}^{\infty} b_n/n$ is convergent.

[Hint: Use Theorem 11.3.2.]

11.6. Show that the trigonometric series, $\sum_{n=2}^{\infty} (\sin nx)/\log n$, is not a Fourier series of any integrable function.

[Hint: If it were a Fourier series of a function $f(x)$, then $b_n = 1/\log n$ would be the Fourier coefficient of an odd function. Apply now part (b) of Exercise 11.5 and show that this assumption leads to a contradiction.]

11.7. Consider the Fourier series of $f(x) = x$ given in Example 11.2.1.

(a) Show that

$$\frac{x^2}{4} = \frac{\pi^2}{12} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos nx}{n^2} \quad \text{for } -\pi < x < \pi.$$

[Hint: Consider the Fourier series of $\int_0^x f(t) dt$.]

(b) Deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

11.8. Make use of the result in Exercise 11.7 to find the sum of the series $\sum_{n=1}^{\infty} [(-1)^{n+1} \sin nx]/n^3$.

11.9. Show that the Fourier transform of $f(x) = e^{-x^2}$, $-\infty < x < \infty$, is given by

$$F(w) = \frac{1}{2\sqrt{\pi}} e^{-w^2/4}.$$

[Hint: Show that $F'(w) + \frac{1}{2}wF(w) = 0$.]

11.10. Prove Theorem 11.6.4 using Fubini's theorem.

11.11. Use the Fourier transform to solve the integral equation

$$\int_{-\infty}^{\infty} f(x-y)f(y) dy = e^{-x^2/2}$$

for the function $f(x)$.

- 11.12.** Consider the function $f(x) = x$, $-\pi < x < \pi$, with $f(-\pi) = f(\pi) = 0$, and $f(x)$ 2π -periodic defined on $(-\infty, \infty)$. The Fourier series of $f(x)$ is

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1} \sin nx}{n}.$$

Let $s_n(x)$ be the n th partial sum of this series, that is,

$$s_n(x) = \sum_{k=1}^n \frac{2(-1)^{k+1}}{k} \sin kx.$$

Let $x_n = \pi - \pi/n$.

(a) Show that

$$s_n(x_n) = \sum_{k=1}^n \frac{2 \sin(k\pi/n)}{k}.$$

(b) Show that

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n(x_n) &= 2 \int_0^{\pi} \frac{\sin x}{x} dx \\ &\approx 1.18\pi. \end{aligned}$$

Note: As $n \rightarrow \infty$, $x_n \rightarrow \pi^-$. Hence, for n sufficiently large,

$$s_n(x_n) - f(x_n) \approx 1.18\pi - \pi = 0.18\pi.$$

Thus, near $x = \pi$ [a point of discontinuity for $f(x)$], the partial sums of the Fourier series exceed the value of this function by approximately the amount $0.18\pi = 0.565$. This illustrates the so-called *Gibbs phenomenon* according to which the Fourier series of $f(x)$ “overshoots” the value of $f(x)$ in a small neighborhood to the left of the point of discontinuity of $f(x)$. It can also be shown that in a small neighborhood to the right of $x = -\pi$, the Fourier series of $f(x)$ “undershoots” the value of $f(x)$.

In Statistics

- 11.13.** In the following table, two observations of the resistance in ohms are recorded at each of six equally spaced locations on the perimeter of a

new type of solid circular coil (see Kupper, 1972, Table 1):

ϕ (radians)	Resistance (ohms)
0	13.62, 14.40
$\pi/3$	10.552, 10.602
$2\pi/3$	2.196, 3.696
π	6.39, 7.25
$4\pi/3$	8.854, 10.684
$5\pi/3$	5.408, 8.488

- (a) Use the method of least squares to estimate the parameters in the following trigonometric polynomial of order 2:

$$\eta = \alpha_0 + \sum_{n=1}^2 [\alpha_n \cos n\phi + \beta_n \sin n\phi],$$

where $0 \leq \phi \leq 2\pi$, and η denotes the average resistance at location ϕ .

- (b) Use the prediction equation obtained in part (a) to determine the points of minimum and maximum resistance on the perimeter of the circular coil.

11.14. Consider the following circular data set in which ϕ is wind direction and y is temperature (see Anderson–Cook, 2000, Table 1).

ϕ (radians)	y (°F)	ϕ (radians)	y (°F)
4.36	52	4.54	38
3.67	41	2.62	40
4.36	41	2.97	49
1.57	31	4.01	48
3.67	53	4.19	37
3.67	47	5.59	37
6.11	43	5.59	33
5.93	43	3.32	47
0.52	41	3.67	51
3.67	46	1.22	42
3.67	48	4.54	53
3.32	52	4.19	46
4.89	43	3.49	51
3.14	46	4.71	39

Fit a second – order trigonometric polynomial to this data set, and verify that the prediction equation is given by

$$\hat{y} = 41.33 - 2.43 \cos \phi - 2.60 \sin \phi + 3.05 \cos 2\phi + 2.98 \sin 2\phi.$$

- 11.15.** Let $\{Y_n\}_{n=1}^\infty$ be a sequence of independent, identically distributed random variables with mean μ and variance σ^2 . Let

$$s_n^* = \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}},$$

where $\bar{Y}_n = (1/n)\sum_{i=1}^n Y_i$.

- (a) Find the characteristic function of s_n^* .
(b) Use Theorem 11.7.3 and part (a) to show that the limiting distribution of s_n^* as $n \rightarrow \infty$ is the standard normal distribution.

Note: Part (b) represents the statement of the well-known *central limit theorem*, which asserts that for large n , the arithmetic mean $\bar{Y}_n$ of a sample of independent, identically distributed random variables is approximately normally distributed with mean μ and standard deviation $\sigma/\sqrt{n}$.